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Extensions of simple modules over Leavitt path algebras ☆

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Let E be a directed graph, K any field, and let $L_K(E)$ denote the Leavitt path algebra of E with coefficients in K. For each rational infinite path c^∞ of E we explicitly construct a projective resolution of the corresponding Chen simple left $L_K(E)$-module $V_{[c^\infty]}$. Further, when E is row-finite, for each irrational infinite path p of E we explicitly construct a projective resolution of the corresponding Chen simple left $L_K(E)$-module $V_{[p]}$. For Chen simple modules S, T we describe $\mathrm{Ext}^1_{L_K(E)}(S,T)$ by presenting an explicit K-basis. For any graph E containing at least one cycle, this

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Chen simple module

description guarantees the existence of indecomposable left $L_K(E)$-modules of any prescribed finite length.

1. Introduction

Given any directed graph E and field K, one may construct the *Leavitt path algebra of E with coefficients in K* (denoted $L_K(E)$), as first described in [2] and [3]. Since their introduction, various structural properties of the algebras $L_K(E)$ have been discovered, with a significant number of the results in the subject taking on the following form: $L_K(E)$ has some specified algebraic property if and only if E has some specified graph-theoretic property. (The structure of the field K often plays no role in results of this type.) A few (of many) examples of such results include a description of those Leavitt path algebras which are simple; purely infinite simple; finite dimensional; prime; primitive; exchange; etc.

Although there are graphs for which the structure of corresponding Leavitt path algebra is relatively pedestrian (e.g., is a direct sum of matrix rings either over K, or over the Laurent polynomial algebra $K[x, x^{-1}]$, or some combination thereof), the less-mundane examples of Leavitt path algebras exhibit somewhat exotic behavior. For instance, the prototypical Leavitt path algebra $A = L_K(R_n)$ $(n \geq 2)$, which arises from the graph R_n having one vertex and n loops, has the property that $A \cong A^n$ as left (or right) A-modules. Analogous "super decomposability" properties are also found in other important classes of Leavitt path algebras. These types of structural properties lead to a dearth (if not outright absence) of indecomposable one-sided $L_K(E)$-ideals, which subsequently makes the search for simple (and, more generally, indecomposable) modules over Leavitt path algebras somewhat of a challenge.

For a graph E, an *infinite path in E* is a sequence of edges $e_1e_2e_3\cdots$, for which $s(e_{i+1}) = r(e_i)$ for all $i \in \mathbb{N}$. In [6], Chen produces, for each infinite path p in E, a simple left $L_K(E)$-module $V_{[p]}$. Further, Chen describes, for each sink vertex w of E, a simple left $L_K(E)$-module $\mathcal{N}_w$.

In Section 2 we produce explicit projective resolutions for Chen simple modules. As a result, we will see in Theorem 2.8 that $V_{[c^\infty]}$ is finitely presented for any closed path c. Further, in Theorem 2.20 we give necessary and sufficient conditions on a row-finite graph E which ensure that $V_{[p]}$ is not finitely presented for an irrational infinite path p. In Section 3 we describe the extension groups $\mathrm{Ext}^1(S, T)$ corresponding to any pair of Chen simple modules S, T. Using some general results about uniserial modules over hereditary rings, we conclude by showing (Corollary 3.25) how our description of $\mathrm{Ext}^1(S, S)$ guarantees the existence of indecomposable $L_K(E)$-modules of any prescribed finite length.

We set some notation. A (directed) graph $E = (E^0, E^1, s, r)$ consists of a *vertex set* E^0, an *edge set* E^1, and *source* and *range* functions $s, r : E^1 \to E^0$. For $v \in E^0$, the set of edges $\{e \in E^1 \mid s(e) = v\}$ is denoted $s^{-1}(v)$. E is called *finite* in case both E^0 and E^1

are finite sets. E is called *row-finite* in case $s^{-1}(v)$ is finite for every $v \in E^0$. A *path* α in E is a sequence $e_1 e_2 \cdots e_n$ of edges in E for which $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$. We say that such α has *length* n, and we write $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We view each vertex $v \in E^0$ as a path of length 0, and denote $v = s(v) = r(v)$. We denote the set of paths in E by $\mathrm{Path}(E)$. A path $\sigma = e_1 e_2 \cdots e_n$ in E is *closed* in case $r(e_n) = s(e_1)$. Following [6] (but not standard in the literature), a closed path σ is called *simple* in case $\sigma \neq \beta^m$ for any closed path β and integer $m \geq 2$. A *sink* in E is a vertex $w \in E^0$ for which the set $s^{-1}(w)$ is empty, while an *infinite emitter* in E is a vertex $u \in E^0$ for which the set $s^{-1}(u)$ is infinite.

For any field K and graph E the Leavitt path algebra $L_K(E)$ has been the focus of sustained investigation since 2004. We give here a basic description of $L_K(E)$; for additional information, see [2] or [1]. Let K be a field, and let $E = (E^0, E^1, s, r)$ be a directed graph with vertex set E^0 and edge set E^1. The *Leavitt path K-algebra $L_K(E)$ of E with coefficients in K* is the K-algebra generated by a set $\{v \mid v \in E^0\}$, together with a set of symbols $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(V) $vu = \delta_{v,u} v$ for all $v, u \in E^0$,
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$,
(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$, and
(CK2) $v = \sum_{\{e \in E^1 \mid s(e) = v\}} ee^*$ for every $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

An alternate description of $L_K(E)$ may be given as follows. For any graph E let $\widehat{E}$ denote the “double graph” of E, gotten by adding to E an edge e^* in a reversed direction for each edge $e \in E^1$. Then $L_K(E)$ is the usual path algebra $K\widehat{E}$, modulo the ideal generated by the relations (CK1) and (CK2).

It is easy to show that $L_K(E)$ is unital if and only if $|E^0|$ is finite; in this case, $1_{L_K(E)} = \sum_{v \in E^0} v$. Every element of $L_K(E)$ may be written as $\sum_{i=1}^n k_i \alpha_i \beta_i^*$, where k_i is a nonzero element of K, and each of the α_i and β_i are paths in E. If $\alpha \in \mathrm{Path}(E)$ then we may view $\alpha \in L_K(E)$, and will often refer to such α as a *real path* in $L_K(E)$; analogously, for $\beta = e_1 e_2 \cdots e_n \in \mathrm{Path}(E)$ we often refer to the element $\beta^* = e_n^* \cdots e_2^* e_1^*$ of $L_K(E)$ as a *ghost path* in $L_K(E)$. The map $KE \to L_K(E)$ given by the K-linear extension of $\alpha \mapsto \alpha$ (for $\alpha \in \mathrm{Path}(E)$) is an injection of K-algebras by [1, Corollary 1.5.12].

The ideas presented in the following few paragraphs come from [6]; however, some of the notation we use here differs from that used in [6], in order to make our presentation more notationally consistent with the general body of literature regarding Leavitt path algebras.

Let p be an *infinite path in E*; that is, p is a sequence $e_1 e_2 e_3 \cdots$, where $e_i \in E^1$ for all $i \in \mathbb{N}$, and for which $s(e_{i+1}) = r(e_i)$ for all $i \in \mathbb{N}$. We emphasize that while the phrase *infinite path in E* might seem to suggest otherwise, an infinite path in E is not an element of $\mathrm{Path}(E)$, nor may it be interpreted as an element of the path algebra KE

nor of the Leavitt path algebra $L_K(E)$. (Such a path is sometimes called a *left*-infinite path in the literature.) We denote the set of infinite paths in E by E^∞.

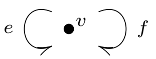
For $p = e_1e_2e_3 \cdots \in E^\infty$ and $n \in \mathbb{N}$ we denote by $\tau_{\leq n}(p)$, or often more efficiently by p_n, the (finite) path $e_1e_2 \cdots e_n$, while we denote by $\tau_{>n}(p)$ the infinite path $e_{n+1}e_{n+2}\cdots$. We note that $\tau_{\leq n}(p)$ is an element of $\mathrm{Path}(E)$ (and thus may be viewed as an element of $L_K(E)$), and that p is the concatenation $p = \tau_{\leq n}(p) \cdot \tau_{>n}(p)$.

Let c be a closed path in E. Then the path $ccc\cdots$ is an infinite path in E, which we denote by c^∞. We call an infinite path of the form c^∞ a *cyclic infinite* path. For c a closed path in E let d be the simple closed path in E for which $c = d^n$. Then $c^\infty = d^\infty$ as elements of E^∞.

If p and q are infinite paths in E, we say that p and q are *tail equivalent* (written $p \sim q$) in case there exist integers m, n for which $\tau_{>m}(p) = \tau_{>n}(q)$; intuitively, $p \sim q$ in case p and q eventually become the same infinite path. Clearly $\sim$ is an equivalence relation on E^∞, and we let $[p]$ denote the $\sim$ equivalence class of the infinite path p.

The infinite path p is called *rational* in case $p \sim c^\infty$ for some closed path c. By a previous observation, we may assume without loss of generality that such c is a simple closed path. In particular, for any closed path c we have that c^∞ is rational. If $p \in E^\infty$ is not rational we say p is *irrational.*

Example 1.1. Let R_2 denote the “rose with two petals” graph



Then $q = efeffefffeffffe\cdots$ is an irrational infinite path in R_2^∞. Indeed, it is easy to show that there are uncountably many distinct irrational infinite paths in R_2^∞. We note additionally that there are infinitely many simple closed paths in $\mathrm{Path}(R_2)$, for instance, any path of the form ef^i for $i \in \mathbb{Z}^+$.

Let M be a left $L_K(E)$-module. For each $m \in M$ we define the $L_K(E)$-homomorphism $\hat{\rho}_m : L_K(E) \to M$, given by $\hat{\rho}_m(r) = rm$. The restriction of the right-multiplication map $\hat{\rho}_m$ may also be viewed as an $L_K(E)$-homomorphism from any left ideal I of $L_K(E)$ into M. When $I = L_K(E)v$ for some vertex v of E, we will denote $\hat{\rho}_m$ simply by ρ_m.

Following [6], for any infinite path p in E we construct a simple left $L_K(E)$-module $V_{[p]}$, as follows.

Definition 1.2. Let p be an infinite path in the graph E, and let K be any field. Let $V_{[p]}$ denote the K-vector space having basis $[p]$, that is, having basis consisting of distinct elements of E^∞ which are tail-equivalent to p. For any $v \in E^0$, $e \in E^1$, and $q = f_1f_2f_3\cdots \in [p]$, define

$$v \cdot q = \begin{cases} q & \text{if } v = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \qquad e \cdot q = \begin{cases} eq & \text{if } r(e) = s(f_1) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

$$e^* \cdot q = \begin{cases} \tau_{>1}(q) & \text{if } e = f_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the K-linear extension to all of $V_{[p]}$ of this action endows $V_{[p]}$ with the structure of a left $L_K(E)$-module.

Theorem 1.3. *(See [6, Theorem 3.3].) Let E be any directed graph and K any field. Let $p \in E^\infty$. Then the left $L_K(E)$-module $V_{[p]}$ described in Definition 1.2 is simple. Moreover, if $p, q \in E^\infty$, then $V_{[p]} \cong V_{[q]}$ as left $L_K(E)$-modules if and only if $p \sim q$, which happens precisely when $V_{[p]} = V_{[q]}$.*

We will refer to a module of the form $V_{[p]}$ as presented in Theorem 1.3 as a *Chen simple module.*

For any sink w in a graph E, Chen in [6] presents a construction, similar in flavor to the one given in Definition 1.2, of a simple left $L_K(E)$-module $\mathcal{N}_w$. He then shows that $\mathcal{N}_w$ is isomorphic as a left $L_K(E)$-module to the left ideal $L_K(E)w$ of $L_K(E)$ generated by w. Observe that, for any sink w, the ideal $L_K(E)w$ is spanned by the paths in E ending in w. Moreover for any $i \in \mathbb{Z}^+$, we get that $w^i = w$ and thus we can consider $w = w^\infty$ as an element in E^∞. For these reasons, for any sink w of E, we refer to $\mathcal{N}_w = L_K(E)w$ as a Chen simple module and, for consistency, we denote $\mathcal{N}_w$ by $V_{[w^\infty]}$.

Remark 1.4. By invoking a powerful result of Bergman, it was established in [3, Theorem 3.5] that, when E is row-finite, then $L_K(E)$ is hereditary, i.e., every left ideal of $L_K(E)$ is projective. This presumably could make the search for projective resolutions of various $L_K(E)$-modules somewhat easier, in that the projectivity of left ideals is already a given. However, much of the strength of our results lies in our explicit description of the kernels of germane maps; for instance, it is these explicit descriptions which will allow us to analyze the Ext^1 groups of the Chen simple modules.

A significant majority of the structural properties of a Leavitt path algebras $L_K(E)$ do not rely on the specific structure of the field K. The results contained in this article are no exceptions. So while each of the statements of the results made herein should also contain the explicit hypothesis "Let K be any field", we suppress this statement throughout for efficiency. For a field K, $K^\times$ denotes the nonzero elements of K.

2. Projective resolutions of Chen simple modules over $L_K(E)$

The goal of this section is to present an explicit description of a projective resolution of S, where S is a Chen simple module over the Leavitt path algebra $L_K(E)$. Such an explicit description will provide a strengthening of some previously established results (see [4, Proposition 4.1]), as well as provide the necessary foundation for subsequent results. As we shall see, the description of projective resolutions, as well as the description of the Ext^1 groups, of Chen simple modules will proceed based on which of the following three types describes the module:

(1) $V_{[w^\infty]} \cong L_K(E)w$ where w is a sink,
(2) $V_{[c^\infty]}$ where c is a simple closed path;
(3) $V_{[q]}$ where q is an irrational infinite path.

Let v be any vertex in E. Since v is an idempotent, the left ideal $L_K(E)v$ is a projective left $L_K(E)$-module. Therefore projective resolutions of Chen simple modules of type 1 are easy:

Proposition 2.1 *(Type (1)). Let w be a sink in E. Then the Chen simple left module $V_{[w^\infty]}$ is projective.*

Proof. We have $V_{[w^\infty]} \cong L_K(E)w$ as left $L_K(E)$-modules by [6]. $\square$

We now begin the process of describing projective resolutions of Chen simple modules of the second type, namely, of the form $V_{[c^\infty]}$ for c a simple closed path.

Notations. Let $c = e_1e_2 \cdots e_t$ be a simple closed path in E, with $v = s(e_1) = r(e_t)$.

(1) For $0 \le i \le t$ we define $c_i := e_1e_2 \cdots e_i$ and $d_i := e_{i+1}e_{i+2} \cdots e_t$ (where $c_0 = v = d_t$ and $c_t = c = d_0$). Then clearly $c = c_id_i$ for each $0 \le i \le t$.
(2) For $n \ge 0$ we let c^{-n} denote $(c^*)^n$, and let c^0 denote $v = s(c)$.
(3) An element μ of $L_K(E)$ is said to be a *standard form monomial* in case there exist $\alpha, \beta \in \text{Path}(E)$ for which $\mu = \alpha\beta^*$. We denote the set of standard form monomials in $L_K(E)$ by $\mathcal{S}$. For $\mu = \alpha\beta^*$ a standard form monomial we define $r(\mu) := r(\beta^*) = s(\beta)$; that is, $r(\mu)$ is the unique element $v \in E^0$ for which $\mu v = \mu$. Define

$$\mathcal{S}_1(c) := \{\mu \in \mathcal{S} \mid \mu \cdot c^N = 0 \text{ in } L_K(E) \text{ for some } N \in \mathbb{N}\}, \text{ and } \mathcal{S}_2(c) := \mathcal{S} \setminus \mathcal{S}_1(c).$$

Although $\mathcal{S}_1(c)$ and $\mathcal{S}_2(c)$ depend on c, we will often simply write $\mathcal{S}_1$ and $\mathcal{S}_2$ for these sets.

By analyzing the form of monomials in $L_K(E)$, we get the following description of the elements of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Lemma 2.2. *Let $0 \ne \mu \in \mathcal{S}$. If c is a sink then $\mu \in \mathcal{S}_1$ if and only if $r(\mu) \ne c$. If c is a closed path $e_1e_2 \cdots e_t$, then $\mu \in \mathcal{S}_1$ if and only if μ is of one of the following two forms:*

(1) $r(\mu) \ne s(c)$ (i.e., $\mu \cdot s(c) = 0$ in $L_K(E)$), or
(2) $\mu = \mu' f^* c_i^* (c^*)^n$ for some $n, i \in \mathbb{Z}^+$, some $\mu' \in \mathcal{S}$, and some $f \in E^1$ for which $s(f) = s(e_{i+1})$ but $f \ne e_{i+1}$.

Consequently, $0 \ne \mu \in \mathcal{S}_2$ if and only if $\mu = \alpha c_i^* (c^*)^n$ for some path α in E, and some pair of non-negative integers n, i.

Lemma 2.3. *Let c be a closed path in the graph E, and let $v = s(c)$.*

(1) *For any $z \in \mathbb{Z}$ we have $c^z - v \in L_K(E)(c - v)$.*
(2) *Suppose $\mu \in \mathcal{S}_1(c)$. Then $\mu \in (\sum_{u\in E^0\setminus\{v\}} L_K(E)u) \bigcup L_K(E)(c - v)$.*

Proof. (1) If $z = 0$ we have $s(c) - v = 0 = 0(c - v)$. For $z > 0$ we have $c^z - v = (c^{z-1} + c^{z-2} + \cdots + c + v)(c - v)$. For $z < 0$ we have $c^z - v = -c^z(c^{-z} - v)$, which is in $L_K(E)(c - v)$ by the previous case.

(2) Suppose $\mu \in \mathcal{S}_1(c)$. If $r(\mu) \neq v$ then $\mu \in \sum_{u\in E^0\setminus\{v\}} L_K(E)u$. On the other hand, suppose $r(\mu) = v$, and that $\mu \cdot c^N = 0$ for some $N \in \mathbb{N}$. But $r(\mu) = v$ gives $\mu v = \mu$, so that with the hypothesis $\mu = -\mu(c^N - v)$, which gives that $\mu \in L_K(E)(c^N - v) \subseteq L_K(E)(c-v)$ by the previous paragraph. $\square$

Remark 2.4. Let $p = e_1e_2\cdots$ be an infinite path in E. If $p = \tau_{>r}(p)$ for some $r > 0$, then p is a rational path of the form $p = c^\infty$, where c is the closed path $e_1e_2\cdots e_r$. This follows from the observation that $p = \tau_{>r}(p)$ implies $p = \tau_{>ir}(p)$ for all $i \in \mathbb{N}$.

Lemma 2.5. *Let c be a simple closed path $e_1e_2\cdots e_t$ in E with $s(c) = r(c) = v$. Suppose α and β are paths in E for which $0 \neq \alpha \cdot c^\infty = \beta \cdot c^\infty$ in $V_{[c^\infty]}$. Then there exists $N \in \mathbb{Z}^+$ for which $\alpha = \beta c^N$ or $\beta = \alpha c^N$.*

Consequently, $\alpha \cdot c^\infty = \beta \cdot c^\infty$ in $V_{[c^\infty]}$ implies $\alpha - \beta \in L_K(E)(c - v)$.

Proof. Assume $\alpha = f_1f_2\cdots f_\ell$ and $\beta = g_1g_2\cdots g_m$, where the f_i and g_j are edges in E. If $\ell = m$, from $\alpha \cdot c^\infty = \beta \cdot c^\infty$ we get $\alpha = \beta$. So assume $m > \ell$; we have

$$g_1\cdots g_\ell g_{\ell+1}\cdots g_m = f_1\cdots f_\ell c^n e_1e_2\cdots e_k$$

with $m - \ell = k + n \times t$, $k \leq t$. If $k < t$, from $\alpha \cdot c^\infty = \beta \cdot c^\infty$ we get

$$c^\infty = c_{k+1}\cdots c_t \cdot c^\infty = \tau_{>k}(c^\infty);$$

then by Remark 2.4 we would have $c^\infty = (e_1\cdots e_k)^\infty$, a contradiction since c is simple. Therefore $k = t$ and $\beta = \alpha c^{n+1}$. The case $m < \ell$ is identical.

For the second statement, note that $0 \neq \alpha \cdot c^\infty = \beta \cdot c^\infty$ gives $r(\alpha) = r(\beta)$; denote this common vertex by v. So if $\alpha = \beta c^n$ then $\alpha - \beta = \beta(c^n - v)$, which is in $L_K(E)(c - v)$ by Lemma 2.3(1). The case $\beta = \alpha c^n$ is identical. $\square$

Proposition 2.6. *Let E be any graph. Let c be a simple closed path in E, and let v denote $s(c) = r(c)$. Let $\rho_{c^\infty} : L_K(E)v \to V_{[c^\infty]}$ and $\hat{\rho}_{c^\infty} : L_K(E) \to V_{[c^\infty]}$ denote the right multiplication by c^∞. Then*

$$\mathrm{Ker}(\rho_{c^\infty}) = L_K(E)(c - v) \quad \textit{and} \quad \mathrm{Ker}(\hat{\rho}_{c^\infty}) = \left(\sum_{u\in E^0\setminus\{v\}} L_K(E)u\right) \oplus L_K(E)(c - v).$$

Proof. Since $(c-v)\cdot c^\infty = c^\infty - c^\infty = 0$ in $V_{[c^\infty]}$, we get $L_K(E)(c-v) \subseteq \mathrm{Ker}(\rho_{c^\infty})$. We now proceed to show that $\mathrm{Ker}(\rho_{c^\infty}) \subseteq L_K(E)(c-v)$. For notational convenience we denote the left ideal $L_K(E)(c-v)$ of $L_K(E)$ by J.

So let $\lambda \in \mathrm{Ker}(\rho_{c^\infty})$, and write

$$\lambda = \sum_{\mu\in\mathcal{M}} k_\mu \mu$$

where $\mathcal{M} \subseteq \mathcal{S}$ is some finite set of distinct standard form monomials in $L_K(E)$, and $k_\mu \in K^\times$. By Lemma 2.3 we may assume that $\mathcal{M} \subseteq \mathcal{S}_2$; that is, by Lemma 2.2, we may assume that, for each $\mu \in \mathcal{M}$, $\mu = \alpha_\mu c_{i_\mu}^*(c^*)^{n_\mu}$ for some path α_μ, some $0 \le i_\mu \le t$, and some $n_\mu \ge 0$.

So we have $\lambda = \sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*(c^*)^{n_\mu}$. By hypothesis $\lambda\cdot c^\infty = 0$ in $V_{[c^\infty]}$, so that

$$\sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*(c^*)^{n_\mu}\cdot c^\infty = 0 \ \text{ in } V_{[c^\infty]}.$$

But $(c^*)^n\cdot c^\infty = c^\infty$ in $V_{[c^\infty]}$ for any $n\in\mathbb{Z}$. So

$$\sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*\cdot c^\infty = 0 \ \text{ in } V_{[c^\infty]}.$$

Also, $c_i^*\cdot c^\infty = d_i\cdot c^\infty$ in $V_{[c^\infty]}$ for any $0\le i\le t$. So

$$\sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu d_{i_\mu}\cdot c^\infty = 0 \ \text{ in } V_{[c^\infty]}.$$

Now define

$$\lambda' = \sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu d_{i_\mu}.$$

Then the previous equation gives that $\lambda' \in \mathrm{Ker}(\rho_{c^\infty})$.

We claim that $\lambda \in J$ if and only if $\lambda' \in J$. To show this, we show that $\bar\lambda = \overline{\lambda'}$ as elements of $L_K(E)/J$. We note first that $\overline{c_i^*} = \overline{d_i}$ in $L_K(E)/J$; this follows immediately from the observation that $d_i - c_i^* = c_i^*(c-v) \in J$. But then in $L_K(E)/J$ we have

$$\begin{aligned}\bar\lambda &= \overline{\sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*(c^*)^{n_\mu}}\\ &= \sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*\overline{(c^*)^{n_\mu}}\\ &= \sum_{\mu\in\mathcal{M}} k_\mu\alpha_\mu c_{i_\mu}^*\bar v \qquad \text{by Lemma 2.3(1)}\end{aligned}$$

$$= \sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu \overline{c^*_{i_\mu}}$$

$$= \sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu \overline{d_{i_\mu}} \qquad \text{by the above note}$$

$$= \overline{\sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu d_{i_\mu}}$$

$$= \overline{\lambda'}.$$

Thus in order to show that $\lambda \in J$, it suffices to show that $\lambda' \in J$, i.e., that $\overline{\lambda'} = \bar{0}$ in $L_K(E)/J$. But $\lambda' \in \operatorname{Ker}(\rho_{c^\infty})$, i.e., $\sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu d_{i_\mu} \cdot c^\infty = 0$ in $V_{[c^\infty]}$. Now partition $\mathcal{M} = \sqcup_{t=1}^{\ell} \mathcal{M}_t$ in such a way that $\mu \sim \mu' \in \mathcal{M}_t$ (for some t) if and only if $\alpha_\mu d_{i_\mu} \cdot c^\infty = \alpha_{\mu'} d_{i_{\mu'}} \cdot c^\infty$ in $V_{[c^\infty]}$.

By Lemma 2.5, if $\mu \sim \mu'$ then $\overline{\alpha_\mu d_{i_\mu}} = \overline{\alpha_{\mu'} d_{i_{\mu'}}}$ in $L_K(E)/J$; we denote this common element of $L_K(E)/J$ by $\overline{x_t}$.

Now $\sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu d_{i_\mu} \cdot c^\infty = 0$ gives

$$\sum_{t=1}^{\ell} \sum_{\mu \in \mathcal{M}_t} k_\mu \alpha_\mu d_{i_\mu} \cdot c^\infty = 0,$$

which by the linear independence of sets of distinct elements of the form $\alpha \cdot c^\infty$ in $V_{[c^\infty]}$ gives $\sum_{\mu \in \mathcal{M}_t} k_\mu = 0$ for each $1 \le t \le \ell$. But then

$$\begin{aligned}
\overline{\lambda'} &= \overline{\sum_{\mu \in \mathcal{M}} k_\mu \alpha_\mu d_{i_\mu}} = \overline{\sum_{t=1}^{\ell} \sum_{\mu \in \mathcal{M}_t} k_\mu \alpha_\mu d_{i_\mu}} \\
&= \sum_{t=1}^{\ell} \sum_{\mu \in \mathcal{M}_t} k_\mu \overline{\alpha_\mu d_{i_\mu}} = \sum_{t=1}^{\ell} \sum_{\mu \in \mathcal{M}_t} k_\mu \overline{x_t} \\
&= \sum_{t=1}^{\ell} (\sum_{\mu \in \mathcal{M}_t} k_\mu) \overline{x_t} = \sum_{t=1}^{\ell} (0) \overline{x_t} = \bar{0},
\end{aligned}$$

which establishes that $\operatorname{Ker}(\rho_{c^\infty}) \subseteq L_K(E)(c-v)$, as desired. The claim about $\hat{\rho}_{c^\infty}$ follows easily from $L_K(E) = \sum_{u \in E^0 \setminus \{v\}} L_K(E)u \oplus L_K(E)v$. $\square$

Lemma 2.7. *Let E be any graph. Let c be a simple closed path in E based at the vertex v, and let $r \in L_K(E)v$. Then $r(c-v) = 0$ in $L_K(E)$ if and only if $r = 0$. In particular, the map*

$$\rho_{c-v} : L_K(E)v \to L_K(E)(c-v)$$

is an isomorphism of left $L_K(E)$-modules.

Furthermore, if E is a finite graph, then the map

$$\hat{\rho}_{c-1} : L_K(E) \to L_K(E)(c-1)$$

is an isomorphism of left $L_K(E)$-modules.

Proof. Let $r \in L_K(E)v$. If $r(c-v) = 0$ then $rc = rv = r$, which recursively gives $rc^j = r$ for any $j \geq 1$. Now write $r = \sum_{i=1}^{n} k_i \alpha_i {\beta_i}^*$, where the α_i and β_i are in Path(E). We note that, for any $m \in \mathbb{N}$, if $\beta \in$ Path(E) has length at most m, then β^* has the property that $\beta^* c^m$ is either 0 or an element of Path(E) in $L_K(E)$. Now let N be the maximum length of the paths in the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$. Then the above discussion shows that rc^N is an element of $L_K(E)v$ of the form $\sum_{i=1}^{n} k_i \gamma_i$, where $\gamma_i \in$ Path(E) for $1 \leq i \leq n$; that is, $rc^N \in KE$. But $rc^N = r$, so that $r \in KE$. However, the equation $rc = r$ (i.e., $r(c-v) = 0$) has only the zero solution in KE by a degree argument. So $r = 0$.

The second statement is established in an almost identical manner. □

We now have all the tools to describe a projective resolution for the modules $V_{[c^\infty]}$ where c is a simple closed path, thus completing the study of the second type of Chen simple module.

Theorem 2.8 *(Type(2)). Let E be any graph. Let c be a simple closed path in E, with $v = s(c)$. Then the Chen simple module $V_{[c^\infty]}$ is finitely presented. Indeed, a projective resolution of $V_{[c^\infty]}$ is given by*

$$0 \longrightarrow L_K(E)v \xrightarrow{\rho_{c-v}} L_K(E)v \xrightarrow{\rho_{c^\infty}} V_{[c^\infty]} \longrightarrow 0\ .$$

If E is a finite graph, an alternate projective resolution of $V_{[c^\infty]}$ is given by

$$0 \longrightarrow L_K(E) \xrightarrow{\hat{\rho}_{c-1}} L_K(E) \xrightarrow{\hat{\rho}_{c^\infty}} V_{[c^\infty]} \longrightarrow 0\ .$$

Proof. $V_{[c^\infty]}$ is a simple left $L_K(E)$-module by Theorem 1.3, and $c^\infty = vc^\infty$ is a nonzero element in $V_{[c^\infty]}$. So the map $\rho_{c^\infty} : L_K(E)v \to V_{[c^\infty]}$ is surjective. By Proposition 2.6 we have $\mathrm{Ker}(\rho_{c^\infty}) = L_K(E)(c-v)$. We get the first short exact sequence since by Lemma 2.7 the map

$$\rho_{c-v} : L_K(E)v \to L_K(E)(c-v)$$

is an isomorphism of left $L_K(E)$-modules. Moreover since v is idempotent, $L_K(E)v$ is a projective left $L_K(E)$-module.

Assume now that E is a finite graph. Let us see that $\mathrm{Ker}(\hat{\rho}_{c^\infty}) = L_K(E)(c-1)$. Since $\mathrm{Ker}(\hat{\rho}_{c^\infty})$ clearly contains u for any $u \neq v \in E^0$, we have $c - 1 = c - \sum_{u\in E^0} u = (c-v) - \sum_{u\neq v} u \in \mathrm{Ker}(\hat{\rho}_{c^\infty})$. But for any $u \neq v = s(c)$ we have $uc = 0$, so that

$u = -u(c-1) \in L_K(E)(c-1)$. Since $c-v = v(c-1) \in L_K(E)(c-1)$, using Proposition 2.6 we have shown that each of the generators of $\mathrm{Ker}(\hat{\rho}_{c^\infty})$ is in $L_K(E)(c-1)$. But by Lemma 2.7, $\hat{\rho}_{c-1} : L_K(E) \to L_K(E)(c-1)$ is an isomorphism of left $L_K(E)$-modules, thus establishing the result. $\square$

Corollary 2.9. *Let E be any graph. Let c be a simple closed path in E, with $v = s(c)$. Then the Chen simple module $V_{[c^\infty]}$ has projective dimension* 1*.*

Proof. From Theorem 2.8 we get the exact sequence

$$0 \longrightarrow L_K(E)v \xrightarrow{\rho_{c-v}} L_K(E)v \xrightarrow{\rho_{c^\infty}} V_{[c^\infty]} \longrightarrow 0\ .$$

Since v is an idempotent in $L_K(E)$, the left module $L_K(E)v$ is projective and hence $V_{[c^\infty]}$ has projective dimension ≤ 1. The left module $V_{[c^\infty]}$ is not projective, otherwise the above sequence splits and $L_K(E)v$ would contain a direct summand isomorphic to $V_{[c^\infty]}$; in particular $L_K(E)v$ would contain a nonzero element α (the element corresponding to c^∞) such that $c\alpha = \alpha$ and hence $c^n\alpha = \alpha$ for each $n \in \mathbb{N}$. This is impossible by a degree argument. $\square$

Before we present a projective resolution of the third type of Chen simple module, we study, in the situation where E is row-finite, right multiplication by any of the monomial generators of $V_{[c^\infty]}$ for c a simple closed path or a sink. We first introduce some notation which will be useful throughout the remainder of the section.

Definition 2.10. Let E be any graph. Let $\beta = e_1e_2\cdots e_n$ be a path in E. For each $1 \leq i \leq n$ let β_i denote $e_1e_2\cdots e_i$. For each $0 \leq i \leq n-1$ let

$$X_i(\beta) = \{f \in E^1 \mid s(f) = s(e_{i+1}),\ \text{and}\ f \neq e_{i+1}\}.$$

The elements of $X_i(\beta)$ are called the *exits* of β at $s(e_{i+1})$. Note that, for a given i, it is possible that $X_i(\beta) = \emptyset$. For each $i \geq 0$ let $J_i(\beta)$ be the left ideal of $L_K(E)$ defined by setting

$$J_i(\beta) = \sum_{f \in X_i(\beta)} L_K(E) f^* \beta_i^*.$$

(So possibly $J_i(\beta) = \{0\}$, precisely when $X_i(\beta) = \emptyset$.) When the path β is clear from context, we may denote $X_i(\beta)$ (resp., $J_i(\beta)$) by X_i (resp., J_i).

Now let $p = e_1e_2e_3\cdots \in E^\infty$ be an infinite path in E. Let p_0 denote $s(e_1)$, and for each $i \geq 0$ let p_{i+1} denote $\tau_{\leq i+1}(p) = e_1e_2\cdots e_{i+1}$. For each $i \geq 0$ we define

$$X_i(p) := X_i(p_{i+1}),\ \text{and}\ J_i(p) := J_i(p_{i+1}).$$

Definition 2.11. Let E be any graph. Let $\beta = e_1 e_2 \cdots e_n$ be a path in E for which no vertex of β is an infinite emitter. For $0 \leq i \leq n-1$ let

$$F_i(\beta) = \sum_{f \in X_i(\beta)} f f^* \in L_K(E).$$

Note that this sum is finite by the hypothesis on β. (We interpret $F_i(\beta)$ as 0 in case $X_i(\beta) = \emptyset$.) In particular, by the (CK2) relation we have

$$s(e_{i+1}) - F_i(\beta) = e_{i+1} e_{i+1}^*$$

for $0 \leq i \leq n-1$, and by (CK1) that $F_i(\beta) e_{i+1} = 0$.

Lemma 2.12. *Let E be any graph. Let $\alpha = e_1 e_2 \cdots e_n$ be a path in E for which no vertex of α is an infinite emitter. Let α_i denote $e_1 e_2 \cdots e_i$ for each $1 \leq i \leq n$ (so in particular $\alpha = \alpha_n$). Suppose $q, x \in L_K(E)$ satisfy the equation $q\alpha = x$ in $L_K(E)$. Then*

$$q = x\alpha^* + q\alpha_{n-1} F_{n-1}(\alpha)\alpha_{n-1}^* + \cdots + q\alpha_1 F_1(\alpha)\alpha_1^* + qF_0(\alpha).$$

Proof. Multiply both sides of the equation $q\alpha = x$ by e_n^*, to get

$$x e_n^* = q\alpha e_n^* = q e_1 \cdots e_{n-1} e_n e_n^* = q e_1 \cdots e_{n-1}(s(e_n) - F_{n-1}(\alpha)).$$

Multiplying the final term and switching sides, this gives

$$q e_1 \cdots e_{n-1} = x e_n^* + q e_1 \cdots e_{n-1} F_{n-1}(\alpha).$$

Multiplying now both sides of this displayed equation on the right by e_{n-1}^*, and proceeding in the same way, we easily get

$$q e_1 \cdots e_{n-2} = x e_n^* e_{n-1}^* + q e_1 \cdots e_{n-1} F_{n-1}(\alpha) e_{n-1}^* + q e_1 \cdots e_{n-2} F_{n-2}(\alpha).$$

Continuing in this way, after n steps we reach

$$q = x e_n^* e_{n-1}^* \cdots e_1^* + q e_1 \cdots e_{n-1} F_{n-1}(\alpha) e_{n-1}^* \cdots e_1^* + \cdots + q e_1 F_1(\alpha) e_1^* + qF_0(\alpha)$$

as desired. □

Of course, if c is a simple closed path or a sink, any nonzero element of the Chen simple module $V_{[c^\infty]}$ generates $V_{[c^\infty]}$; this is in particular true of any "monomial" element αc^∞, where α is a path in E for which $r(\alpha) = s(c)$. We describe here the projective resolution corresponding to such elements, in case E is row-finite.

Theorem 2.13 *(Types (1) & (2)). Let E be any graph. Let c be a simple closed path or a sink in E, with $v = s(c)$. Let $\alpha = e_1e_2\cdots e_n$ be any path in E for which no vertex of α is an infinite emitter, and for which $r(\alpha) = r(e_n) = v$. Let u denote $s(\alpha) = s(e_1)$. Then the following is a projective resolution of the Chen simple $L_K(E)$-module $V_{[c^\infty]}$:*

$$0 \longrightarrow L_K(E)(\alpha c\alpha^* - u) \longrightarrow L_K(E)u \xrightarrow{\rho_{\alpha c^\infty}} V_{[c^\infty]} \longrightarrow 0\ .$$

Proof. Since $u\alpha = \alpha$, we have that $\rho_{\alpha c^\infty}(u) = \alpha c^\infty$ is a nonzero element of the Chen simple module $V_{[c^\infty]}$, so that $\rho_{\alpha c^\infty}$ is surjective. So we need only establish that $\mathrm{Ker}(\rho_{\alpha c^\infty}) = L_K(E)(\alpha c\alpha^* - u)$. Since $\rho_{\alpha c^\infty}(\alpha c\alpha^* - u) = (\alpha c\alpha^* - u)\alpha c^\infty = \alpha cc^\infty - \alpha c^\infty = 0$, it remains only to show that $\mathrm{Ker}(\rho_{\alpha c^\infty}) \subseteq L_K(E)(\alpha c\alpha^* - u)$.

So let $q \in \mathrm{Ker}(\rho_{\alpha c^\infty})$; specifically, $q\alpha c^\infty = 0$. But then $q\alpha \in \mathrm{Ker}(\rho_{c^\infty})$, which, by Theorem 2.8, is precisely $L_K(E)(c - v)$. So

$$q\alpha = r(c - v)$$

for some $r \in L_K(E)$. By Lemma 2.12, we have

$$q = r(c - v)\alpha^* + q\alpha_{n-1}F_{n-1}(\alpha)\alpha_{n-1}^* + \cdots + q\alpha_1F_1(\alpha)\alpha_1^* + qF_0(\alpha).$$

Using this representation of q, it suffices to show that each of the summands on the right hand side is an element of $L_K(E)(\alpha c\alpha^* - u)$. Since easily we get $(c - v)\alpha^* = \alpha^*(\alpha c\alpha^* - u)$, we have that $r(c - v)\alpha^* \in L_K(E)(\alpha c\alpha^* - u)$. But for each $0 \le i \le n - 1$ we have $F_i(\alpha)\alpha_i^*\alpha = F_i(\alpha)e_{i+1}\cdots e_n = 0$ (using the observation made in Definition 2.11). Using this, we see that $q\alpha_iF_i(\alpha)\alpha_i^* = -q\alpha_iF_i(\alpha)\alpha_i^*(\alpha c\alpha^* - u)$, so that $q\alpha_iF_i(\alpha)\alpha_i^* \in L_K(E)(\alpha c\alpha^* - u)$ for each $0 \le i \le n - 1$, thus completing the proof. □

We now describe a projective resolution of the third type of Chen simple module, namely, one corresponding to an irrational infinite path. Whereas a Chen simple corresponding to a rational path is always finitely presented, we will see that the determination of the finite-presentedness of a Chen simple corresponding to an irrational infinite path will depend on the structure of the graph itself.

Lemma 2.14. *Let E be any graph. Let p be an irrational infinite path in E with $s(p) = v$, and let $\rho_p : L_K(E)v \to V_{[p]}$ be the map $r \mapsto rp$. Let $x \in \mathrm{Ker}(\rho_p)$. Then there exists $n_x \in \mathbb{N}$ such that $x\tau_{\le n_x}(p) = 0$ in $L_K(E)$. In other words, if $xp = 0$ in $V_{[p]}$, then $xp_{n_x} = 0$ in $L_K(E)$ for some finite initial segment p_{n_x} of p.*

Proof. Let $x = \sum_{i=1}^m k_i\alpha_i\beta_i^* \in \mathrm{Ker}(\rho_p)$, where $\alpha_i, \beta_i \in \mathrm{Path}(E)$. Denote by N the maximum length of the β_i, $i = 1, \ldots, m$. We have

$$\rho_p(x) = \sum_{i=1}^m k_i\alpha_i\rho_p(\beta_i^*) = \sum_{i=1}^m k_i\alpha_i\rho_{\tau_{>N}(p)}(\beta_i^*\tau_{\le N}(p)).$$

Since the length of each β_i is less than or equal to N, $t_i := \beta_i^* \tau_{\leq N}(p)$ is either zero or a real path. Therefore

$$0 = \rho_p(x) = \sum_{i=1}^{m} k_i \alpha_i \rho_{\tau_{>N}(p)}(t_i) = \rho_{\tau_{>N}(p)}(\sum_{i=1}^{m} k_i \alpha_i t_i) = \rho_{\tau_{>N}(p)}(\sum_{\ell=1}^{m'} h_\ell \gamma_\ell)$$
$$= \sum_{\ell=1}^{m'} h_\ell \gamma_\ell \tau_{>N}(p),$$

where the γ_ℓ ($1 \leq \ell \leq m'$) are distinct elements of the form $\alpha_i t_i$ in $\mathrm{Path}(E)$, and $h_\ell \in K$. Since p is irrational and γ_ℓ ($1 \leq \ell \leq m'$) are distinct paths, we claim that the infinite paths $\gamma_\ell \tau_{>N}(p)$ ($1 \leq \ell \leq m'$) are distinct elements of $V_{[p]}$, as follows. Assume to the contrary that $\gamma_i \tau_{>N}(p) = \gamma_j \tau_{>N}(p)$ for some $i \neq j$; necessarily γ_i and γ_j have distinct lengths s_i and s_j. Assume $s_i - s_j = s > 0$; then

$$\gamma_i \tau_{>N}(p) = \gamma_j \kappa_i \tau_{>N}(p) = \gamma_j \tau_{>N}(p),$$

and hence $\kappa_i \tau_{>N}(p) = \tau_{>N}(p)$, where κ_i is a suitable element of $\mathrm{Path}(E)$ having length s. Therefore $\tau_{>N}(p) = \tau_{>s}(\tau_{>N}(p)) = \tau_{>s+N}(p)$. But this property implies by Remark 2.4 that p is rational, contrary to hypothesis. Thus the $\gamma_\ell \tau_{>N}(p)$ ($1 \leq \ell \leq m'$) are distinct infinite paths.

Consequently, the set $\{\gamma_\ell \tau_{>N}(p) \mid 1 \leq \ell \leq m'\}$ is linearly independent over K, so the previously displayed equation $0 = \sum_{\ell=1}^{m'} h_\ell \gamma_\ell \tau_{>N}(p)$ yields that $h_\ell = 0$ for each $1 \leq \ell \leq m'$. Therefore

$$x\tau_{\leq N}(p) = \sum_{i=1}^{m} k_i \alpha_i \beta_i^* \tau_{\leq N}(p) = \sum_{i=1}^{m} k_i \alpha_i t_i = \sum_{\ell=1}^{m'} h_\ell \gamma_\ell = 0,$$

as desired. □

Lemma 2.15. *Let E be any graph. Suppose β is a path of length n in E for which no vertex of β is an infinite emitter, and for which $s(\beta) = v$. For each $0 \leq i \leq n-1$ let $J_i(\beta)$ be the left ideal of $L_K(E)$ given in Definition 2.10. If $x \in L_K(E)v$ has $x\beta = 0$, then $x \in \sum_{i=0}^{n-1} J_i(\beta)$.*

Proof. Write $\beta = e_1 e_2 \cdots e_n$. For each $1 \leq i \leq n$ let $\beta_i = e_1 e_2 \cdots e_i$. So $\beta = \beta_n$, and thus by hypothesis we are assuming that $x\beta_n = 0$. Then using the (CK2) relation at the vertices $s(e_1), s(e_2), \cdots, s(e_n)$ in order (this is possible by the hypothesis on β), and interpreting empty sums as 0, we get

$$x = xv = x(\sum_{f \in X_0(\beta)} ff^* + \beta_1 \beta_1^*) = x(\sum_{f \in X_0(\beta)} ff^*) + x\beta_1 r(\beta_1) \beta_1^*$$

$$
\begin{aligned}
&= x(\sum_{f\in X_0(\beta)} ff^*) + x\beta_1(\sum_{f\in X_1(\beta)} ff^* + e_2e_2^*)\beta_1^* \\
&= x(\sum_{f\in X_0(\beta)} ff^*) + x\beta_1(\sum_{f\in X_1(\beta)} ff^*)\beta_1^* + x\beta_2 r(\beta_2)\beta_2^* \\
&= \cdots \\
&= x(\sum_{f\in X_0(\beta)} ff^*) + x\beta_1(\sum_{f\in X_1(\beta)} ff^*)\beta_1^* + \cdots + x\beta_{n-1}(\sum_{f\in X_{n-1}(\beta)} ff^*)\beta_{n-1}^* \\
&\quad + x\beta_{n-1}e_ne_n^*\beta_{n-1}^* \\
&= x(\sum_{f\in X_0(\beta)} ff^*) + x\beta_1(\sum_{f\in X_1(\beta)} ff^*)\beta_1^* + \cdots + x\beta_{n-1}(\sum_{f\in X_{n-1}(\beta)} ff^*)\beta_{n-1}^* + x\beta_n\beta_n^* \\
&= x(\sum_{f\in X_0(\beta)} ff^*) + x\beta_1(\sum_{f\in X_1(\beta)} ff^*)\beta_1^* + \cdots + x\beta_{n-1}(\sum_{f\in X_{n-1}(\beta)} ff^*)\beta_{n-1}^* + 0,
\end{aligned}
$$

with the final statement following from the hypothesis that $x\beta = x\beta_n = 0$. Thus

$$
\begin{aligned}
x = &\sum_{f\in X_0(\beta)} (xf)f^* + \sum_{f\in X_1(\beta)} (x\beta_1 f)f^*\beta_1^* + \cdots + \sum_{f\in X_{n-1}(\beta)} (x\beta_{n-1}f)f^*\beta_{n-1}^* \\
&\in \sum_{i=0}^{n-1} J_i(\beta). \qquad \square
\end{aligned}
$$

Lemma 2.16. *Let E be a finite graph, and let $p = e_1e_2\cdots \in E^\infty$ be an irrational infinite path in E. Then $X_i(p)$ is nonempty for infinitely many $i \in \mathbb{Z}^+$. Consequently, in this case, $J_i(p)$ is nonzero for infinitely many $i \in \mathbb{Z}^+$.*

Proof. Suppose to the contrary that there exists $N \in \mathbb{N}$ for which $X_i(p) = \emptyset$ for all $i \geq N$. Since E^0 is finite, there exist $t, t' \geq N$, $t < t'$, for which $s(e_t) = s(e_{t'})$. But $X_t(p) = \emptyset$ then gives $e_t = e_{t'}$, and in a similar manner yields $e_{t+\ell} = e_{t'+\ell}$ for all $\ell \in \mathbb{Z}^+$. If d denotes the closed path $e_te_{t+1}\cdots e_{t'-1}$, then we get $p \sim d^\infty$, the desired contradiction. $\square$

We note that Lemma 2.16 is not necessarily true without the finiteness hypothesis on the graph. For instance, let $M_{\mathbb{N}}$ be the graph

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \xrightarrow{e_3} \cdots$$

and let $p \in M_{\mathbb{N}}^\infty$ be the irrational infinite path $e_1e_2\cdots$. Then $X_i(p) = \emptyset$ for all $i \geq 0$.

Corollary 2.17. *Let E be any graph. Let $p \in E^\infty$ be an irrational infinite path in E for which no vertex of p is an infinite emitter, and for which $s(p) = v$. Let*

$\rho_p : L_K(E)v \to V_{[p]}$ be the map $r \mapsto rp$. For each $i \geq 0$ let $J_i(p)$ be the left ideal of $L_K(E)$ given in Definition 2.10. Then

$$\operatorname{Ker}(\rho_p) = \bigoplus\nolimits_{i=0}^{\infty} J_i(p).$$

Proof. Clearly, for every $i \geq 0$, each element of $J_i(p)$ is in $\operatorname{Ker}(\rho_p)$. Now suppose $x \in L_K(E)$ has $xp = 0$ in $V_{[p]}$. By Lemma 2.14, $x\beta = 0$ where $\beta = p_n = \tau_{\leq n}(p)$ for some $n \in \mathbb{N}$. Then Lemma 2.15, together with the definition of $J_i(p)$ for $p \in E^\infty$, gives that $\operatorname{Ker}(\rho_p) = \sum_{i=0}^{\infty} J_i(p)$.

Now suppose $\sum_{i=0}^{n} r_i = 0$ in $L_K(E)$, where $r_i \in J_i(p)$ for $0 \leq i \leq n$. By construction, $r_i p_n = 0$ for all $i < n$. On the other hand, for any $f \in X_n(p)$, $f^* p_n^* p_n p_n^* = f^* p_n^*$, so that $r_n p_n p_n^* = r_n$ for all $r_n \in J_n(p)$. Thus multiplying both sides of the proposed equation $\sum_{i=0}^{n} r_i = 0$ on the right by $p_n p_n^*$ gives $r_n = 0$. Using this same idea iteratively, we get $r_i = 0$ for all $0 \leq i \leq n$, so that the sum is indeed direct. $\square$

We note that Corollary 2.17 is not necessarily true without the finite emitter hypothesis on the vertices of p. For instance, let F be the graph

$$\bullet^w \xLeftarrow{\ \infty\ } \bullet^v \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \xrightarrow{e_3} \cdots$$

where there are infinitely many edges $\{f_i \mid i \in \mathbb{Z}^+\}$ from v to w. Let p be the irrational infinite path $e_1 e_2 \cdots$, and let $\rho_p : L_K(E)v \to V_{[p]}$ as usual. Then easily $x = v - e_1 e_1^* \in \operatorname{Ker}(\rho_p)$. However, $x \notin \sum_{i=0}^{\infty} J_i(p)$, since otherwise this would yield that v is a finite sum of $e_1 e_1^*$ plus terms of the form $r_i f_i f_i^*$ for $r_i \in L_K(E)$, which cannot happen as v is an infinite emitter.

Remark 2.18. Corollary 2.17 shows that if E is row-finite and p is an irrational infinite path, then $\operatorname{Ker}(\rho_p)$ is generated by those ghost paths of $L_K(E)$ which annihilate some (finite) initial path of p. Effectively, this is the main difference between the rational and irrational cases; in the rational case, where $p = d^\infty$ and $s(p) = v$, there are additional elements in $\operatorname{Ker}(\rho_p)$ which are not of this form, namely, elements of the form $r(d - v)$ where $r \in L_K(E)$.

Lemma 2.19. *Let p be an irrational infinite path in an arbitrary graph E. For $f \in E^1$ let v_f denote the vertex $r(f)$. Then, for each $i \geq 0$,*

$$J_i(p) = \bigoplus_{f \in X_i(p)} L_K(E) f^* p_i^* \cong \bigoplus_{f \in X_i(p)} L_K(E) v_f,$$

as left $L_K(E)$-modules. In particular, each $J_i(p)$ is a projective left $L_K(E)$-module.

Proof. By definition $J_i(p) = \sum_{f \in X_i(p)} L_K(E) f^* p_i^*$. We claim the sum is direct. So suppose $0 = \sum_{f \in X_i(p)} r_f f^* p_i^*$, with $r_f \in L_K(E)$ for each $f \in X_i(p)$. Without loss we may assume that each expression $r_f f^*$ is nonzero, so that we may further assume without loss that $r_f v_f = r_f$ for each $f \in X_i(p)$. Take $g \in X_i(p)$; by multiplying $0 = \sum_{f \in X_i(p)} r_f f^* p_i^*$ on the right by $p_i g$, and using the (CK1) relation, we get $0 = r_g g^* g = r_g \cdot v_g = r_g$. Thus the sum is direct, so that $J_i(p) = \oplus_{f \in X_i(p)} L_K(E) f^* p_i^*$. But for $g \in X_i(p)$ it is easy to show that $L_K(E) g^* p_i^* \cong L_K(E) v_g$, by the map $x \mapsto x p_i g$. □

Theorem 2.20 *(Type (3)). Let E be any graph. Let $p \in E^\infty$ be an irrational infinite path in E for which no vertex of p is an infinite emitter. Then the Chen simple $L_K(E)$-module $V_{[p]}$ is finitely presented if and only if $X_i(p)$ is nonempty only for finitely many $i \in \mathbb{Z}^+$.*

In particular, if E is a finite graph, then $V_{[p]}$ is not finitely presented.

Proof. Let v denote $s(p)$. We consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\rho_p) \longrightarrow L_K(E)v \xrightarrow{\rho_p} V_{[p]} \longrightarrow 0 \ .$$

By Corollary 2.17 we have that $\mathrm{Ker}(\rho_p) = \oplus_{i=0}^\infty J_i(p)$. Furthermore, each $J_i(p)$ is projective by Lemma 2.19, so the given exact sequence is a projective resolution of $V_{[p]}$. Therefore $V_{[p]}$ is finitely presented if and only if $J_i(p)$ is nonzero only for finitely many $i \in \mathbb{Z}^+$, i.e. $X_i(p)$ is nonempty only for finitely many $i \in \mathbb{Z}^+$.

For the particular case, when E is finite then by Lemma 2.16 $J_i(p)$ is nonzero for infinitely many i. □

Corollary 2.21. *If E is a finite graph, and $p \in E^\infty$ is an irrational infinite path in E, then the Chen simple $L_K(E)$-module $V_{[p]}$ has projective dimension 1.*

Proof. From Corollary 2.17 we get the exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^\infty J_i(p) \longrightarrow L_K(E)v \xrightarrow{\rho_p} V_{[p]} \longrightarrow 0 \ .$$

Since v is an idempotent in $L_K(E)$, the left module $L_K(E)v$ is projective; by Lemma 2.19 also $\bigoplus_{i=0}^\infty J_i(p)$ is projective and hence $V_{[p]}$ has projective dimension ≤ 1. Since E is finite, $J_i(p)$ is not zero for infinitely many i and hence $\bigoplus_{i=0}^\infty J_i(p)$ is not finitely generated. Then the left module $V_{[p]}$ is not projective, otherwise $\bigoplus_{i=0}^\infty J_i(p)$ would be a not finitely generated direct summand of a cyclic module: contradiction. □

Remark 2.22. Let $M_{\mathbb{N}}$ be the graph

$$\bullet \xrightarrow{e_1} \bullet \xrightarrow{e_2} \bullet \xrightarrow{e_3} \cdots$$

considered previously, and let $p \in M_{\mathbb{N}}^\infty$ be the irrational infinite path $e_1 e_2 e_3 \cdots$. Then $X_i(p) = \emptyset$ for all $i \geq 0$. So by Corollary 2.17, the Chen simple module $V_{[p]}$ is isomorphic to $L_K(E)v$, and hence it is projective.

Example 2.23. We reconsider the graph R_2 and irrational infinite path $q = efeffefffe\cdots \in R_2^\infty$ described in Example 1.1. Then, as R_2 is finite, Theorem 2.20 yields that the Chen simple module $V_{[q]}$ is not finitely presented.

Remark 2.24. We note that Theorems 2.8 and 2.20 strengthen and sharpen [4, Proposition 4.1], most notably because we have been able to explicitly describe a projective resolution of each of the Chen simple modules.

In [5, Theorem 4.12] it is shown that for any graph E, for any vertex $v \in E^0$, $L(E)v$ is a simple left ideal if and only if v is a *line point*, i.e. in the full subgraph of E generated by $\{u \in E^0 \mid \text{there is a path from } v \text{ to } u\}$ there are no cycles, and there are no vertices which emit more than one edge. Our results allow us to recover [5, Theorem 4.12], as follows.

Corollary 2.25. *Let E be any graph. Let $u \in E^0$. Then $L_K(E)u$ is simple if and only if u is a line point.*

Proof. There are three possibilities:

(1) there is a path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = u$ and for which $r(\alpha) = w$ is a sink in E;
(2) there is a path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = u$ and for which $r(\alpha) = v$ is the source of a simple closed path c;
(3) there is an infinite irrational path q for which $s(q) = u$.

If in α (cases 1 and 2) or in q (case 3) there is an infinite emitter x, then $L_K(E)x$ is not a simple submodule of $L_K(E)v$ (see [5, Lemma 4.3]). Therefore we can assume that α (cases 1 and 2) and q (case 3) have no infinite emitter.

Cases 1 and 2. By Theorem 2.13 $L_K(E)u$ is a simple module if and only if $\alpha c\alpha^* = u$, where c is either a simple closed path or a sink. If c is a simple closed path, by a degree argument $\alpha c\alpha^*$ is not a vertex. If c is a sink and $\alpha = e_1\cdots e_\ell$ then $\alpha c\alpha^* = \alpha\alpha^* = e_1\cdots e_\ell e_\ell^*\cdots e_1^*$ is equal to u if and only if $e_ie_i^* = s(e_i)$ for $i = 1,\dots,\ell$, i.e. if and only if $s(e_i)$ is the source of only one edge, i.e. u is a line point.

Case 3. By Corollary 2.17, $L_K(E)u$ is simple if and only if $\bigoplus_{i=0}^\infty J_i(p) = 0$ and the latter is equivalent to p having no exits, i.e. u is a line point. $\square$

3. Extensions of Chen simple modules

In this section we use the results of Section 2 to describe $\mathrm{Ext}^1_{L_K(E)}(S,T)$, where S and T are Chen simple modules over the Leavitt path algebra $L_K(E)$ corresponding to a finite graph E. As a consequence, this will allow us to (among other things) construct classes of indecomposable non-simple $L_K(E)$-modules.

We give here a short review of Ext^1; see e.g. [8] for more information. Let R be a ring, and let M, N be left R-modules. Suppose

$$0 \longrightarrow Q \xrightarrow{\mu} P \xrightarrow{f} M \longrightarrow 0$$

is a short exact sequence with P projective. Then there is an exact sequence of abelian groups

$$\mathrm{Hom}_R(P,N) \xrightarrow{\mu_*} \mathrm{Hom}_R(Q,N) \xrightarrow{\Delta_f} \mathrm{Ext}^1_R(M,N) \longrightarrow 0,$$

where $\mu_*(\varphi) = \varphi \circ \mu$ for $\varphi \in \mathrm{Hom}_R(P,N)$, and Δ_f is the "connecting morphism". If μ is viewed as an inclusion of submodules, then $\mu_*(\varphi) = \varphi|_Q$, the restriction of φ to Q. Exactness yields that $\mathrm{Ext}^1_R(M,N) = 0$ if and only μ_* is surjective. Moreover, $\mathrm{Ext}^1_R(M,N) \neq 0$ if and only if there exists a non-splitting short exact sequence

$$0 \to N \to L \to M \to 0,$$

i.e., L is a non-trivial extension of N by M. For instance if M, N are simple left R-modules, then $\mathrm{Ext}^1_R(M,N) \neq 0$ if and only if there exist indecomposable left R-modules of length 2 which are extensions of N by M. Finally, observe that if R is a K-algebra over a field K, then the abelian group $\mathrm{Ext}^1_R(M,N)$ has a natural structure of K-vector space for any left R-module M and N.

We outline our approach. There are three types of Chen simple modules: those of the form $V_{[w^\infty]}$ for a sink w; of the form $V_{[c^\infty]}$ for a simple closed path c; and of the form $V_{[p]}$ for an irrational infinite path p. Let T denote any Chen simple module. In Lemma 3.1 we make the (trivial) observation that $\mathrm{Ext}^1_{L_K(E)}(V_{[w^\infty]}, T) = 0$; in Theorem 3.13 we describe $\mathrm{Ext}^1_{L_K(E)}(V_{[c^\infty]}, T)$; and in Theorem 3.21 we describe $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T)$. We recall that we are assuming $w = w^\infty \in E^\infty$ for any sink w.

Lemma 3.1 *(Type (1)). Let E be any graph. Let w be a sink in E, and let T denote any left $L_K(E)$-module. Then $\mathrm{Ext}^1_{L_K(E)}(V_{[w^\infty]}, T) = 0$, i.e. any extension of $V_{[w^\infty]}$ by T splits.*

Proof. This follows immediately from the fact that $V_{[w]} \cong L_K(E)w$ is a projective $L_K(E)$-module (see Proposition 2.1). $\square$

Definition 3.2. Let T be a Chen simple module. Denote by $U(T)$ the set

$$U(T) := \{v \in E^0 \mid vT \neq \{0\}\} = \{v \in E^0 \mid \text{there exists } t \in T \text{ with } vt \neq 0\}.$$

Remark 3.3. Let T be a Chen simple module and let $q = e_1 e_2 \cdots \in E^\infty$ such that $T = V_{[q]}$. Then $U(T)$ consists of those vertices v for which there is a path $\alpha \in \mathrm{Path}(E)$ having $s(\alpha) = v$ and $r(\alpha) = s(e_i)$ for some $i \geq 1$. Equivalently, a vertex $v \in U(T)$ if and only if there is an infinite path tail-equivalent to q starting from v. Hence $U(T)$ is a feature of the Chen simple module T that can be read directly from the graph E.

Definitions 3.4. Let E be any graph and let d be a simple closed path in E.

For any $p \in E^\infty$ we say that p is *divisible by* d if $p = dp'$ for some $p' \in E^\infty$.

For any $q \in E^\infty$, we define the set

$$L_{(d,q)} := \{p \in E^\infty \mid p \sim q,\ s(p) = s(d),\ \text{and } p \text{ is not divisible by } d\} \subseteq V_{[q]},$$

where $V_{[q]}$ is the Chen simple $L_K(E)$-module generated by q.

An infinite path p is divisible by a simple closed path $d \in E$ if and only if $d = t_{\leq \ell}(p)$, where ℓ is the length of d. The set $L_{(d,q)}$ consists of those infinite paths which start at $s(d)$, and which eventually equal some tail of q, but do not start out by traversing the closed path d. Observe that the subset $L_{(d,q)}$ of $V_{[q]}$ does not depend on q but only on the equivalence class $[q]$. Let $T = V_{[q]}$; if q is not tail equivalent to d^∞, then there exists $q' \sim q$ such that $d \nmid q'$ and hence T has a generator not divisible by d.

Remark 3.5. Let d be a simple closed path in E and $q \in E^\infty$.

(1) Suppose q is not tail equivalent to d^∞ and consider the Chen simple module $V_{[q]}$; we can assume without loss of generality that q is not divisible by d. The set $L_{(d,q)}$ is not empty if and only if $s(d)$ belongs to $U(V_{[q]})$; in such a case any $0 \neq t \in V_{[q]}$ for which $s(d)t = t$ is a linear combination of infinite paths tail equivalent to q whose sources coincide with $s(d)$. In particular, taking in account the divisibility by d of these infinite paths, t can be written in a unique way as

$$t = t_0 + dt_1 + d^2 t_2 + \cdots + d^s t_s,$$

where the t_i are K-linear combinations of elements in $L_{(d,q)}$ and $t_s \neq 0$. We call $s \geq 0$ the d**-degree** of t and we denote it by $\deg_d(t)$.

(2) Suppose $q = d^\infty$. Then $L_{(d,d^\infty)} \neq \emptyset$ if and only if there exists a cycle $c \neq d$ with $s(c) = s(d)$. Any $0 \neq t \in V_{[d^\infty]}$ for which $s(d)t = t$ can be written in a unique way as

$$t = kd^\infty + t_0 + dt_1 + d^2 t_2 + \cdots + d^s t_s,$$

where the $t_i \in V_{[d^\infty]}$ are K-linear combinations of elements in $L_{(d,d^\infty)}$ and $t_s \neq 0$. We call $s \geq 0$ the d**-degree** of t and we denote it by $\deg_d(t)$.

In particular, any $0 \neq t \in L_{(d,q)}$ has d-degree equal to 0. We emphasize that, in case $q = d^\infty$, the d-degree of the element d^∞ of $V_{[d^\infty]}$ is zero too: $\deg_d(d^\infty) = 0$. The d-degree is not defined on 0.

Example 3.6. We revisit the graph R_2 given by

$$\begin{tikzcd} \bullet^{v} \arrow[loop left, "e"] \arrow[loop right, "f"] \end{tikzcd}.$$

Consider the simple closed path e and the rational infinite path f^∞. Then $L_{(e,f^\infty)} = \{p \in R_2^\infty \mid p \sim f^\infty \text{ and } p \text{ is not divisible by } e\} \subseteq V_{[f^\infty]}$ contains, for instance, the infinite paths $\{f^i e^j f^\infty \mid i \geq 1, j \geq 0\}$. (There are additional elements of $L_{(e,f^\infty)}$, for instance, $fefef^\infty$.) Moreover, consider an element of the form $e^j f^i ef^\infty \in V_{[f^\infty]}$, with $i \geq 1$ and $j \geq 0$. Then $\deg_e(e^j f^i ef^\infty) = j$.

On the other hand, $L_{(f,f^\infty)} = \{p \in R_2^\infty \mid p \sim f^\infty$ and p is not divisible by $f\}$ contains the infinite paths $\{e^i f^\infty \mid i \geq 1\}$. Note that the element f^∞ of V_{f^∞} is defined to have $\deg_f(f^\infty) = 0$.

Recall that $L_K(E)$ is a ring wit unity if and only if E is finite.

Lemma 3.7. *Let E be a finite graph. Let d be a simple closed path and $q \in E^\infty$. Let $t \in V_{[q]}$, and consider the equation in the variable X*

$$(d-1)X = t.$$

The equation admits a solution in $V_{[q]}$ if one of the following holds:

(1) *$s(d)t = 0$;*
(2) *$t = d^n p - p$ for some $p \in L_{(d,q)}$ and $n \geq 0$.*

Proof. (1) is easy, since if $s(d)t = 0$ then $dt = 0$, and hence $X = -t$ is a solution. (2) is nearly as easy, since we have $(d-1)\sum_{i=0}^{n-1} d^i p = d^n p - p = t$, and hence $X = \sum_{i=0}^{n-1} d^i p$ is a solution. $\square$

Lemma 3.8. *Let E be a finite graph. Let d be a simple closed path and let $q \in E^\infty$. Assume either $q = d^\infty$ or q is a generator of $V_{[q]}$ not divisible by d. Let $0 \neq t \in V_{[q]}$, and consider the equation*

$$(d-1)X = t.$$

Assume $t = d^n t'$ for some $n \geq 0$ and some $0 \neq t' \in V_{[q]}$ for which $s(d)t' = t'$ and $\deg_d(t') = 0$. Then the equation has no solution in $V_{[q]}$. In particular:

(1) *the equation $(d-1)X = t$ has no solution in $V_{[q]}$ whenever $t \in L_{(d,q)}$, and*
(2) *the equation $(d-1)X = d^\infty$ has no solution in $V_{[d^\infty]}$.*

Proof. Let $v = s(d)$. Since $vd = d$ and $t = d^n t'$, we get $vt = t$. So if x is a solution of $(d-1)X = t$, then we would have $v(d-1)x = t$, so that $(d-1)vx = t$; thus we may assume without loss that $vx = x$. Hence the equation yields

$$x = dx - t,$$

and, since $t \neq 0$, necessarily then $x \neq 0$. Let $\deg_d(x) = s$ and write

$$x = kd^\infty + x_0 + dx_1 + \cdots + d^s x_s,$$

where the x_is are linear combination of elements in $L_{(d,q)}$, and $k = 0$ in case $q \neq d^\infty$. Then, using $d \cdot d^\infty = d^\infty$, we get

$$\begin{aligned} t = d^n t' &= (d-1)x = dx - x \\ &= kd^\infty - kd^\infty - x_0 + d(x_0 - x_1) + \cdots + d^s(x_{s-1} - x_s) + d^{s+1}x_s \\ &= -x_0 + d(x_0 - x_1) + \cdots + d^s(x_{s-1} - x_s) + d^{s+1}x_s. \end{aligned}$$

We claim that this is impossible. Set $x_{-1} = 0 = x_{s+1}$. By the uniqueness of the decomposition in Remark 3.5, since $\deg_d(t') = 0$, one gets $t' = x_{n-1} - x_n$ and $x_{i-1} - x_i = 0$ for any $i \neq n$, $-1 \leq n \leq s+1$. Then we have $0 = x_0 = x_1 = \cdots = x_{n-1}$ and $t' = -x_n = \cdots - x_{s+1} = 0$, contradiction. □

Assume E is a finite graph and d a simple closed path in E. In order to compute the groups $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$ for any Chen simple module T, we can consider the projective resolution of $V_{[d^\infty]}$

$$0 \to L_K(E) \xrightarrow{\hat{\rho}_{(d-1)}} L_K(E) \xrightarrow{\hat{\rho}_{d^\infty}} V_{[d^\infty]} \to 0$$

ensured by Theorem 2.8.

Lemma 3.9. *Let E be a finite graph. Let d be a simple closed path in E and let T be a Chen simple module. Consider the exact sequence*

$$\mathrm{Hom}_{L_K(E)}(L_K(E), T) \xrightarrow{\hat{\rho}_{(d-1)_*}} \mathrm{Hom}_{L_K(E)}(L_K(E), T) \xrightarrow{\pi} \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) \longrightarrow 0$$

where $\hat{\rho}_{(d-1)_*}(\phi) = \phi \circ \hat{\rho}_{d-1}$, and π is the connecting homomorphism. Then

$$\pi(\hat{\rho}_t) = 0 \text{ if and only if the equation } (d-1)X = t \text{ has a solution in } T.$$

Consequently, $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) = 0$ *if and only if $(d-1)X = t$ has a solution in T for every $t \in T$.*

Proof. By exactness it follows that $\pi(\hat{\rho}_t) = 0$ if and only if there exists $x \in T$ such that

$$\hat{\rho}_t = \hat{\rho}_{(d-1)_*}(\hat{\rho}_x) = \hat{\rho}_x \circ \hat{\rho}_{(d-1)} = \hat{\rho}_{(d-1)x}$$

i.e. if and only if the equation $(d-1)X = t$ has a solution in T.

The final statement follows directly from the exactness of the displayed sequence. □

Theorem 3.10 *(Type (2)). Let E be a finite graph. Let d be a simple closed path in E and let T be a Chen simple module. Then the following are equivalent:*

(1) $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) \neq 0$.
(2) $s(d) \in U(T)$.

Proof. (1) $\Rightarrow$ (2) If $s(d)T = 0$, then for any $t \in T$ we have $s(d)t = 0$, so by Lemma 3.7(1) the equation $(d-1)X = t$ admits a solution for any $t \in T$. Applying Lemma 3.9(2), we get that $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) = 0$.

(2) $\Rightarrow$ (1) First assume $T \neq V_{[d^\infty]}$. As observed in Remark 3.5, T admits a generator q not divisible by d and $L_{(d,q)}$ is not empty. Let $p \in L_{(d,q)}$. By Lemma 3.8, the equation $(d-1)X = p$ has no solution in $V_{[q]}$ and so, by Lemma 3.9, $\pi(\hat{\rho}_p) \neq 0$.

On the other hand, suppose $T = V_{[d^\infty]}$. By Lemma 3.8, the equation $(d-1)X = d^\infty$ has no solution in $V_{[d^\infty]}$, and so, again invoking Lemma 3.9, $\pi(\hat{\rho}_{d^\infty}) \neq 0$.

In either case we have established the existence of a nonzero element in $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$. $\square$

Corollary 3.11. *Let E be a finite graph. For any simple closed path d, $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]}) \neq 0$.*

Example 3.12. We again revisit the graph R_2:

$$e \circlearrowleft \bullet^v \circlearrowright f \ .$$

Let q be any element of R_2^∞. Let d be any (of the infinitely many) simple closed paths in R_2. Since clearly Condition (2) of Theorem 3.10 is satisfied for $V_{[q]}$, we get that $\mathrm{Ext}^1_{L_K(R_2)}(V_{[d^\infty]}, V_{[q]}) \neq 0$. $\square$

Having now established conditions which ensure that there exist nontrivial extensions of the Chen simple module T by the simple module $V_{[d^\infty]}$, we now give a more explicit description of the number of such extensions.

Proposition 3.13. *Let E be any finite graph. Let d be a simple closed path in E and let T be a Chen simple module. Assume $q \in E^\infty$ such that $T = V_{[q]}$.*

(1) *Suppose $T \neq V_{[d^\infty]}$. Then* $\dim_K \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) = |L_{(d,q)}|$.
(2) *On the other hand,* $\dim_K \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]}) = |L_{(d,d^\infty)}| + 1$.

Proof. We consider the exact sequence

$$\mathrm{Hom}_{L_K(E)}(L_K(E), V_{[q]}) \xrightarrow{\hat{\rho}_{(d-1)*}} \mathrm{Hom}_{L_K(E)}(L_K(E), V_{[q]}) \xrightarrow{\pi} \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[q]}) \longrightarrow 0 \ .$$

(1) Without loss of generality we can assume q is not divisible by d. By Remark 3.5(1) and Theorem 3.10, if $L_{(d,q)} = \emptyset$ then $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T) = 0$. Otherwise, by Lemmas 3.8 and 3.9, $\pi(\hat{\rho}_p) \neq 0$ for any path $p \in L_{(d,q)}$. We claim that the set $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,q)}\}$ is a basis for the vector space $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$. In order to show that $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,q)}\}$ is K-linearly independent, suppose there is a K-linear combination $0 = k_1\pi(\hat{\rho}_{p_1}) + \cdots + k_n\pi(\hat{\rho}_{p_n})$ in $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[q]})$, with $p_i \in L_{(d,q)}$. Let $t = k_1p_1 + \cdots + k_np_n$ in $V_{[q]}$ so that $\pi(\hat{\rho}_t) = 0$ in $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[q]})$. Thus, applying Lemma 3.9, we get that the equation $(d-1)X = t$ has a solution in $V_{[q]}$. If $t \neq 0$, since $s(d)t = t$ and $p_i \in L_{(d,q)}$ we get $\deg_d(t) = 0$, which is a contradiction by Lemma 3.8. Hence $t = 0$ and by the linear independence of $\{p_1, \ldots, p_n\}$ in $V_{[q]}$ we get that $k_1 = \cdots = k_n = 0$. So $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,q)}\}$ is K-linearly independent.

We now show that $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,q)}\}$ spans $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$. As π is surjective, by Lemma 3.9(1) it suffices to show that any $\pi(\hat{\rho}_t) \in \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$ is a K-linear combination of elements from this set. Write $t = t' + t''$ where $t' = \sum_{i=1}^{m_{t'}} k_i p'_i$ with $s(p'_i) = s(d)$ and $t'' = \sum_{j=1}^{m_{t''}} k_j p''_j$ with $s(p''_j) \neq s(d)$. By Lemma 3.7(1), the equation $(d-1)X = t$ has solution in $T = V_{[q]}$ if and only if $(d-1)X = t'$ has solution in $V_{[q]}$, so we can assume without loss of generality that $s(d)t = t$. Hence $t = t_0 + dt_1 + d^2t_2 + \cdots + d^st_s$, and so $\hat{\rho}_t = \hat{\rho}_{t_0} + \hat{\rho}_{dt_1} + \hat{\rho}_{d^2t_2} + \cdots + \hat{\rho}_{d^st_s}$, where each t_i is of the form $t_i = \sum_{j=1}^{m_i} k_{ij}u_{ij}$, for some $u_{ij} \in L(d,q)$. Thus $\pi(\hat{\rho}_t) = \sum_{j=1}^{m_0} k_{0j}\pi(\hat{\rho}_{u_{0j}}) + \sum_{j=1}^{m_1} k_{1j}\pi(\hat{\rho}_{du_{1j}}) + \cdots + \sum_{j=1} m_s k_{sj}\pi(\hat{\rho}_{d^su_{sj}})$. Observe that, by Lemmas 3.7(2) and 3.9, we get $\pi(\hat{\rho}_{d^nu} - \hat{\rho}_u) = 0$ for any $u \in L_{(d,q)}$ and any $n \in \mathbb{N}$, so $\pi(\hat{\rho}_{d^nu}) = \pi(\hat{\rho}_u)$ for any $n \in \mathbb{N}$. Hence $\{\pi(\hat{\rho}_u) \mid u \in L_{(d,q)}\}$ is a set of generators for $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$.

(2) Let us show that $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,d^\infty)}\} \cup \{\pi(\hat{\rho}_{d^\infty})\}$ is a basis for $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]})$. First observe that, by Lemmas 3.8 and 3.9(2), $\pi(\hat{\rho}_{d^\infty}) \neq 0$ and $\pi(\hat{\rho}_p) \neq 0$ for any $p \in L_{(d,d^\infty)}$. Arguing as in part (1) we claim that $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,d^\infty)}\} \cup \{\pi(\hat{\rho}_{d^\infty})\}$ is a linearly independent set in $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]})$. Indeed, consider a K-linear combination $0 = k_0\pi(\hat{\rho}_{d^\infty}) + k_1\pi(\hat{\rho}_{p_1}) + \cdots + k_n\pi(\hat{\rho}_{p_n})$ in $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]})$. Define $y = k_0d^\infty + k_1p_1 + \cdots + k_np_n \in V_{[d^\infty]}$ so that $\pi(\hat{\rho}_y) = 0$ and hence, by Lemma 3.9(2), the equation $(d-1)X = y$ has a solution in $V_{[d^\infty]}$. Note that if $y \neq 0$ then $\deg_d(y) = 0$ (whether or not $k_0 = 0$) since each $p_i \in L_{(d,d^\infty)}$, which is a contradiction by Lemma 3.8. So $y = 0$, which yields that each k_i $(0 \leq i \leq n)$ is 0.

Since any t in $V_{[d^\infty]}$ with $s(d)t = t$ is of the form $t = kd^\infty + t_0 + dt_1 + d^2t_2 + \cdots + d^st_s$, using the same arguments as in part (1) it can be easily be shown that the set $\{\pi(\hat{\rho}_p) \mid p \in L_{(d,d^\infty)}\} \cup \{\pi(\hat{\rho}_{d^\infty})\}$ spans $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, V_{[d^\infty]})$. $\square$

Lemma 3.14. *Let d be a simple closed path in the finite graph E.*

(1) *If $q \in E^\infty$ is irrational and $L_{(d,q)} \neq \emptyset$, then $|L_{(d,q)}|$ is infinite.*
(2) *If $L_{(d,d^\infty)} \neq \emptyset$, then $|L_{(d,d^\infty)}|$ is infinite.*

Proof. (1) Let $q = e_1e_2\cdots$ for $e_i \in E^1$. First notice that, for any $w \in E^0$, if $w = r(e_i)$ for some $i > 0$, then we can assume without loss of generality that $w = r(e_j)$ for infinitely many $j > 0$ (as otherwise, since E^1 is finite, we can replace q with $q' \in E^\infty$ for which $q \sim q'$ and $w \notin (q')^0$).

Consider now an element $p \in L_{(d,q)}$. Then $p = \beta q_0$ and $q = \gamma q_0$ for some $q_0 \in [q]$ and $\beta, \gamma \in \mathrm{Path}(E)$ and β not divisible by d. Consider $w = r(\beta) = s(q_0)$. Then, by the previous assumption, there exists a set of infinite and distinct truncations $\{\tau_{>n_k}(q) \mid k \in \mathbb{N}\}$ such that, for each $k \in \mathbb{N}$, $q = \gamma_k w \tau_{>n_k}(q)$ for some $\gamma_k \in \mathrm{Path}(E)$. Since q is irrational, by Remark 2.4 the infinite paths in the set $\{\tau_{>n_k}(q) \mid k \in \mathbb{N}\}$ are distinct. Hence there are infinitely many distinct elements $\beta\tau_{>n_k}(q)$ in $L_{(d,q)}$, which establishes (1).

(2) If $L_{(d,d^\infty)} \neq \emptyset$, then there is at least one simple closed path c for which $s(c) = s(d)$ and $c \neq d$. Then we easily get that each of the distinct paths $\{c^i d^\infty \mid i \in \mathbb{N}\}$ is tail equivalent to d^∞, which gives that $L_{(d,d^\infty)}$ is infinite. $\square$

Corollary 3.15. *Let d be a simple closed path in E and T a Chen simple module. If $\dim_K \mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$ is finite, then $T = V_{[c^\infty]}$ for a simple closed path c.*

Proof. It follows directly from Lemma 3.14 and Proposition 3.13. $\square$

Example 3.16. For each $n \in \mathbb{N}$, consider the graph

$$E_n \;=\; \begin{tikzcd} \bullet^{v} \arrow[loop left, "d"] \arrow[r, "(n)"] & \bullet^{w} \arrow[loop right, "f"] \end{tikzcd} \;,$$

where the symbol (n) indicates that there are n edges $\{e_1, \ldots, e_n\}$ for which $s(e_i) = v$ and $r(e_i) = w$. Then in E_n we have $L_{(d,f^\infty)} = \{e_1 f^\infty, \ldots, e_n f^\infty\}$, so that $|L_{(d,f^\infty)}| = n$. By Proposition 3.13(1) we conclude that $\dim_K \mathrm{Ext}^1_{L_K(E_n)}(V_{[d^\infty]}, V_{[f^\infty]}) = n$.

Example 3.17. Consider the graph

$$R_1 \;=\; \begin{tikzcd} \bullet^{v} \arrow[loop right, "d"] \end{tikzcd} \;,$$

for which $L_K(R_1) \cong K[x, x^{-1}]$. Then $L_{(d,d^\infty)}$ is empty, hence by Proposition 3.13(2) we conclude that $\dim_K \mathrm{Ext}^1_{L_K(R_1)}(V_{[d^\infty]}, V_{[d^\infty]}) = 1$.

Having given a complete analysis of $\mathrm{Ext}^1_{L_K(E)}(V_{[d^\infty]}, T)$ for any simple closed path d of E and any Chen simple module T, we now analyze $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T)$ for any irrational infinite path p and any Chen simple module T. The projective resolution of $V_{[p]}$ we are going to use is the one introduced in the proof of Theorem 2.20:

$$0 \longrightarrow \mathrm{Ker}(\rho_p) \longrightarrow L_K(E)v \xrightarrow{\rho_p} V_{[p]} \longrightarrow 0 \;,$$

where $\mathrm{Ker}(\rho_p) = \oplus_{i=0}^\infty J_i(p)$ as in Corollary 2.17 and $v = s(p)$.

Remark 3.18. Let E be a finite graph and $u \in L_K(E)$. For each left $L_K(E)$-module M, any morphism $\phi \in \mathrm{Hom}_{L_K(E)}(L_K(E)u, M)$ is the right product by the element $\phi(u)$ of M. If u is an idempotent, then it is $\mathrm{Hom}_{L_K(E)}(L_K(E)u, M) \cong uM$ as abelian groups, by means of the isomorphism $\phi \mapsto \phi(u) = u\phi(u)$.

In order to state the analog of Theorem 3.10 for the irrational case we need the following notation. Let $p = e_1e_2 \cdots \in E^\infty$ be an irrational infinite path in E. For each $i \geq 0$ let $X_i(p)$ denote the set $\{f \in E^1 \mid s(f) = s(e_{i+1}) \text{ and } f \neq e_{i+1}\}$ as presented in Definition 2.10, and define

$$r(X_i(p)) := \{w \in E^0 \mid w = r(f) \text{ for some } f \in X_i(p)\}.$$

Finally, for any $p \in E^\infty$ and any $i \geq 0$ recall from Definitions 2.10 that the left $L_K(E)$-ideal $J_i(p)$ is

$$J_i(p) = \sum_{f \in X_i(p)} L_K(E)f^*p_i^* \ ,$$

where p_i denotes the truncation $\tau_{\leq i}(p)$. As proved in Lemma 2.19, if p is irrational, then

$$J_i(p) \ = \bigoplus_{f \in X_i(p)} L_K(E)f^*p_i^* \ \cong \bigoplus_{f \in X_i(p)} L_K(E)r(f).$$

Lemma 3.19. *Let p be an irrational infinite path in the finite graph E. Let T denote a Chen simple module and let $t \in T$. Then there exists a positive integer $N = N(t)$ for which $(J_i(p))t = 0$ for all $i \geq N$.*

Proof. Assume $q \in E^\infty$ (q can be rational, irrational or a sink) and let $\alpha \in \mathrm{Path}(E)$ with $\mathrm{length}(\alpha) = n$. Observe that, for any $i \geq n$, one has $p_i^*\alpha q = 0$ unless $p_i = \alpha\tau_{\leq i-n}(q)$. So if $p_i^*\alpha q \neq 0$ for all $i \in \mathbb{N}$, we can conclude $p = \alpha q$. Finally notice that, if there exists $N \in \mathbb{N}$ such that $p_N^*\alpha q = 0$, then $p_i^*\alpha q = 0$ for any $i \geq N$.

(Case 1.) Let $T \neq V_{[p]}$ and let $q \in E^\infty$, necessarily not tail-equivalent to p, such that $T = V_{[q]}$. Consider an element $t \in T$. Then t can be written as $t = \sum_{u=1}^{s} k_u\alpha_u\tau_{>i_u}(q)$, where the α_us are in $\mathrm{Path}(E)$ and the i_us are in $\mathbb{N}$, and hence $\alpha_u\tau_{>i_u}(q) \neq p$ for any $u = 1, \dots, s$. Since any element in $(J_i(p))t$ is a finite sum of expressions of the form $k_uf^*p_i^*\alpha_u\tau_{>i_u}(q)$, and since $\alpha_u\tau_{>i_u}(q) \neq p$, by the previous observations we can choose an $N = N(t)$ sufficiently large such that $(J_i(p))t = 0$ for any $i \geq N$.

(Case 2.) On the other hand, let $T = V_{[p]}$ and consider an element $t \in T$. Then t can be written as $t = \sum_{u=1}^{s} k_uq_u$, where the q_us are tail equivalent to p. Let $f \in X_i(p)$, $i \geq 0$; if $q_u = p$ then $f^*p_i^*p = f^*\tau_{>i}(p) = 0$, by construction of f^*. If $q_u \neq p$, there exists an integer $i_u > 0$ such that $\tau_{\leq i_u}(q) \neq p_{i_u}$ and hence $p_{i_u}^*\tau_{\leq i_u}(q) = 0$. Then, by the initial observation, if $N = N_t = \max\{i_u : u = 1, \dots, s\}$, we conclude that $(J_i(p))t = 0$ for any $i \geq N$. $\square$

Lemma 3.20. *Let E be a finite graph, and let p be an infinite irrational path in E. Let T be a Chen simple $L_K(E)$-module. Then $\mathrm{Hom}_{L_K(E)}(J_i(p), T) \neq 0$ if and only if $r(f) \in U(T)$ for some $f \in X_i(p)$.*

Proof. By standard ring theory, we have the following isomorphisms of abelian groups

$$\begin{aligned}\mathrm{Hom}_{L_K(E)}(J_i(p), T) &\cong \mathrm{Hom}_{L_K(E)}(\oplus_{f\in X_i(p)} L_K(E)r(f), T)\\ &\cong \oplus_{f\in X_i(p)}\mathrm{Hom}_{L_K(E)}(L_K(E)r(f), T) \cong \oplus_{f\in X_i(p)} r(f)T,\end{aligned}$$

where the second isomorphism holds because $|X_i(p)|$ is finite, and the final one by Remark 3.18 because each $r(f)$ is idempotent. □

Theorem 3.21 *(Type (3)). Let p be an irrational infinite path in the finite graph E and let T be any Chen simple $L_K(E)$-module. Then $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T) \neq 0$ if and only if $r(X_i(p)) \cap U(T) \neq \emptyset$ for infinitely many $i \geq 0$. In such a situation, $\dim_K(\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T))$ is infinite.*

Proof. ($\Rightarrow$) Suppose $r(X_i(p)) \cap U(T) \neq \emptyset$ for at most finitely many $i \geq 0$. We seek to show that every element of $\mathrm{Hom}_{L_K(E)}(\mathrm{Ker}(\rho_p), T)$ arises as right multiplication by an element of T. We have $\mathrm{Hom}_{L_K(E)}(\mathrm{Ker}(\rho_p), T) = \mathrm{Hom}_{L_K(E)}(\oplus_{i\geq 0} J_i(p), T) \cong \prod_{i\geq 0}\mathrm{Hom}_{L_K(E)}(J_i(p), T)$, which by Lemma 3.20 and hypothesis equals $\prod_{i=0}^N \mathrm{Hom}_{L_K(E)}(J_i(p), T)$ for some $N \in \mathbb{N}$. For each $i \geq 0$ and $f \in X_i(p)$, the element $p_i f f^* p_i^*$ is an idempotent generator of $L_K(E)f^*p_i^*$; moreover $\{p_i f f^* p_i^* : f \in X_i(p), i \geq 0\}$ is a set of orthogonal idempotents in $L_K(E)$. Every element φ of $\mathrm{Hom}_{L_K(E)}(L_K(E)p_i f f^* p_i^*, T)$ is the right multiplication by $\varphi(p_i f f^* p_i^*)$; then every element ψ of $\mathrm{Hom}_{L_K(E)}(\oplus_{i=0}^N J_i(p), T)$ is the right multiplication by $\psi(\sum_{i=0}^N \sum_{f\in X_i(p)} p_i f f^* p_i^*)$. So $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T) = 0$.

($\Leftarrow$) Conversely, let us see that $r(X_i(p)) \cap U(T) \neq \emptyset$ for infinitely many $i \geq 0$ implies that there is an element of $\mathrm{Hom}_{L_K(E)}(\mathrm{Ker}(\rho_p), T)$ which does not arise as a right multiplication by an element of T. By Lemma 3.20 there exists an increasing sequence $(i_n)_{n\in\mathbb{N}}$ of natural numbers such that $\mathrm{Hom}_{L_K(E)}(J_i(p), T) \neq 0$ if and only if $i = i_n$ for a suitable $n \in \mathbb{N}$. Let $\{\phi_i \in \mathrm{Hom}_{L_K(E)}(J_i(p), T) : i \in \mathbb{N}\}$ be a family of morphisms such that $\phi_{i_n} \neq 0$ for each $n \in \mathbb{N}$. Then

$$\varphi = \prod_{i\in\mathbb{N}} \phi_i \in \mathrm{Hom}_{L_K(E)}(\oplus_{i\in\mathbb{N}} J_i(p), T)$$

is a morphism which is not, by Lemma 3.19, right multiplication by element of T.

To establish the final statement, consider an increasing sequence $(i_n)_{n\in\mathbb{N}}$ of natural numbers and a family $\{\phi_{i_n} \in \mathrm{Hom}_{L_K(E)}(J_{i_n}(p), T) : n \in \mathbb{N}\}$ of nonzero morphisms. Define for each prime $z \in \mathbb{N}$ the morphism $\Psi^z \in \mathrm{Hom}_{L_K(E)}(\mathrm{Ker}(\rho_p), T)$ as follows: for each $j \geq 0$,

$$\Psi^z(J_{i_j}(p)) = \psi_{i_j}(J_{i_j}(p)) \text{ if } z \text{ divides } j, \quad \text{while } \Psi^z(J_\ell(p)) = 0 \text{ otherwise.}$$

Observe that $\pi(\Psi^z) \neq 0$, since Ψ^z has infinitely many nonzero components. Finally, $\{\pi(\Psi^z) \mid z \in \mathbb{N},\ z \text{ prime}\}$ is a set of linearly independent elements of $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T))$, as follows. Let F be a finite subset of primes in $\mathbb{N}$, and assume $\sum_{z\in F} k_z \pi(\Psi^z) = 0$; then $\pi(\sum_{z\in F} k_z \Psi^z) = 0$ and therefore $\sum_{z\in F} k_z \Psi^z$ is a right multiplication by an element t of T. By Lemma 3.19 there exists a positive integer $N = N(t)$ for which $(J_i(p))t = 0$ for all $i \geq N$. For each prime $\hat{z} \in F$, let $m_{\hat{z}}$ a natural number greater than N; then

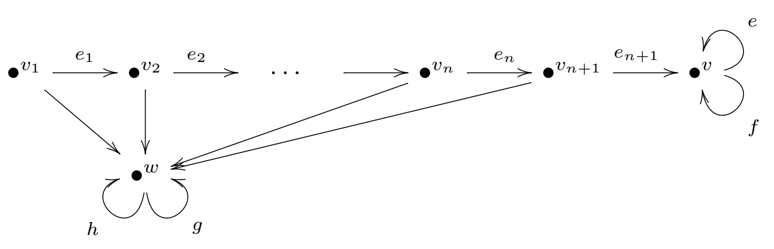
$$0 = (J_{\hat{z}^{m_{\hat{z}}}}(p))t = \sum_{z\in F} k_z \Psi^z (J_{\hat{z}^{m_{\hat{z}}}}(p)) = k_{\hat{z}} \Psi^{\hat{z}} (J_{\hat{z}^{m_{\hat{z}}}}(p))$$

and hence $k_{\hat{z}} = 0$. Hence $\dim_K(\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T))$ is infinite. $\square$

We emphasize the fact that Theorems 3.10 and 3.21 allow us to compute the dimension of the Ext^1-groups between two Chen simple modules completely and solely in terms of properties of the graph E.

Example 3.22. We again revisit the graph R_2 and irrational infinite path $q = efeffefffe\cdots$ of Example 1.1. Let $T = V_{[q]}$. Since clearly $U(T) = \{v\}$ and $r(X_i) = \{v\}$ for all $i \in \mathbb{N}$ as well, Theorem 3.21 yields that $\dim_K(\mathrm{Ext}^1_{L_K(E)}(V_{[q]}, V_{[q]}))$ is infinite.

Example 3.23. With the statement of Theorem 3.21 as motivation, we give examples of graphs E_n, an irrational infinite path p in E_n, and Chen simple $L_K(E_n)$-modules T having $r(X_i(p)) \cap U(T) \neq \emptyset$ for only finitely many $i \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ consider the graph E_n given by



Let p denote the irrational infinite path $e_1e_2\cdots e_ne_{n+1}efeffefffe\cdots$. Let T_1 be the Chen simple $L_K(E)$-module $V_{[g^\infty]}$, and let T_2 denote the Chen simple $L_K(E)$-module $V_{[q]}$ corresponding to the irrational infinite path $q = ghghhghhhg\cdots$. Then for $j = 1, 2$, $r(X_i(p)) \cap U(T_j)$ is nonempty (indeed, equals $\{w\}$) precisely when $0 \leq i \leq n$.

Consequently, by Theorem 3.21, $\mathrm{Ext}^1_{L_K(E)}(V_{[p]}, T_j) = \{0\}$ for $j = 1, 2$.

We conclude the article by demonstrating the existence of indecomposable $L_K(E)$-modules of prescribed finite length, in case E is a finite graph which contains cycles. Recall that a module M is called *uniserial* in case the lattice of submodules of M is totally ordered. In particular, any uniserial module is indecomposable. Moreover, the radical $\mathrm{Rad}(M)$ of a uniserial module M is the unique maximal submodule of M, hence $M/\operatorname{Rad} M$ is simple.

Lemma 3.24. *(See [7, Lemma 16.1 with Proposition 16.2].) Let R be any unital ring. Let U be a uniserial left R-module of finite length, and X a simple left R-module. Consider the morphism $\psi : \mathrm{Ext}^1_R(X,U) \to \mathrm{Ext}^1_R(X, U/\operatorname{Rad} U)$. An extension in $\mathrm{Ext}^1_R(X,U)$ is uniserial if and only if it does not belong to $\operatorname{Ker}\psi$.*

In particular, if R is hereditary, there exists a uniserial extension of U by X if and only if $\mathrm{Ext}^1_R(X, U/\operatorname{Rad} U) \neq 0$.

As observed in Remark 1.4, $L_K(E)$ is hereditary for any row-finite graph E. So Lemma 3.24 gives the following:

Corollary 3.25. *Let E be a finite graph. If S is a Chen simple $L_K(E)$-module such that $\mathrm{Ext}^1_{L_K(E)}(S,S) \neq 0$ and L is a uniserial $L_K(E)$-module such that $L/\operatorname{Rad}(L) \cong S$, then there exists a uniserial $L_K(E)$-module M which is an extension of L by S.*

In particular, for any $n \in \mathbb{N}$ there exists a uniserial $L_K(E)$-module of length n, all of whose composition factors are isomorphic to S.

Proof. The first statement follows directly from Lemma 3.24 and the hereditariness of $L_K(E)$. In order to show the existence of uniserial modules of arbitrary length, first observe that any non-zero element of the abelian group $\mathrm{Ext}^1_{L_K(E)}(S,S) \neq 0$ corresponds to an indecomposable uniserial module L_2 of length 2, with $\mathrm{Rad}(L_2) \cong S$ and $L_2/\mathrm{Rad}(L_2) \cong S$. Then, by applying the first statement, there exists a uniserial module L_3 of length 3 which is an extension of L_2 by S. Since $\mathrm{Rad}(L_3) \cong L_2$ and hence $L_3/\operatorname{Rad}(L_3) \cong S$, we can proceed by induction. $\square$

Observe that if E contains a simple closed path d, by Corollary 3.11 the module $S = V_{[d^\infty]}$ satisfies $\mathrm{Ext}^1_{L_K(E)}(S,S) \neq 0$ and hence Corollary 3.25 applies.

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